Non Quadratic Local Risk-Minimization for Hedging Contingent Claims in the Presence of Transaction Costs

Frédéric Abergel, Nicolas Millot

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Abstract

This paper is devoted to the study of derivative hedging in incomplete markets when frictions are considered. We extend the general local risk minimisation approach introduced in [1] to account for liquidity costs, and derive the corresponding optimal strategies in both the discrete- and continuous-time settings. We examplify our method in the case of stochastic volatility and/or jump-diffusion models.

Introduction

The optimal hedging of derivatives in uncomplete markets is a subject of the utmost importance from a application-driven point of view, as well as an introduction to a host of many challenging mathematical problems. Originating in the pioneering work of [9], the local risk minimization method has been identified as one of the most intuitive and practical way of defining realistic hedging strategies and related option prices. In the original approach developped in [9], the local risk was defined as the second moment of the incremental cost between two consecutive re-hedging periods. In a recent work [1], this approach was revisited so as to extend it to general, convex local risk functionals, and the corresponding optimality conditions were derived in the discrete- and continuous-time settings. This article is devoted to the task of extending our previous results to the realistically important case of transaction costs. More precisely, and contrarily to one of the early and important contributions in that direction, see [14], we do not introduce a bid-ask spread, as this would lead to infinite costs in the continuous limit, but rather, consider as in [5] a supply curve corresponding to the existence of finite liquidity at a given price. Such a paradigm is especially well-suited to the situation of a trader hedging a large book or trading in an illiquid market. It also connects with recent researches on orderbook modelling and market impact, when the supply curve is seen as a smoothed-out version of a stochastic orderbook profile.

The main result of this paper is twofold : on one hand, in a discrete-time setting, the optimality system is fully characterized, and admits a natural interpretation in terms of a non-linear martingale transform orthogonal to the martingale part of a modified price process. Then, extensions to a continuous-time setting are considered, for which only the case of Itō processes can be understood in full generality. For processes having discontinuous paths, pseudo-optimality can be considered, but the connection with the original minimization problem is an open question.

The paper is organized as follows: Section 1 presents, in the discrete-time case, the basic definitions of the cost of a strategy and its associated risk. Section 2 contains the optimality and pseudo-optimality conditions in discrete time, and provide an interpretation of the optimal strategy in terms of a non-linear martingale transform orthogonal to the cost-adjusted price process (the supply price). In Section 3, we extend those results to the continuous-time setting, while Section 4 and 5 are respectively devoted to applications to stochastic volatility and jump-diffusion models.

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1 Liquidity costs and risk process

1.1 Cost process

In this section, we investigate the discrete time setting, studying the existence and uniqueness of solutions to the minimization problem. The market will classically be represented, see e.g. [9][1], by a multi-period model where the risky asset is a strictly positive semimartingale S_k, $(k = 0, \cdots, T)$ on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_k$ denote the $\sigma-$field of events which are observable up to and including time k. We assume that S_k is adapted and square-integrable and that the conditional variance of its returns $\mathbb{E}\left((S_{k+1} - S_k)^2 | \mathcal{F}_k\right) - \mathbb{E}\left(S_{k+1} - S_k | \mathcal{F}_k\right)^2$ is strictly positive P-almost surely. In order to simplify the exposition, the risk-free rate is supposed to be deterministic and therefore, using discounted stock prices, one can assume that it is zero.
The task of interest is that of hedging a contingent claim associated to a square-integrable random variable $H \in L^2(P)$ of the following form $H = \delta^H S_T + \beta^H$, δ^H and β^H being $\mathcal{F}_T-$measurable random variables. We thus consider a general trading strategy Φ represented by two stochastic processes δ_k, $(k = 0, \cdots, T)$ and β_k, $(k = 0, \cdots, T)$, both adapted to $\mathcal{F}_k$ and in $L^2(P)$. δ_k is the amount of stock held during the k^{th} period $= [t_k, t_{k+1})$, and has to be fixed at the beginning of that period. That is to say, δ_k is $\mathcal{F}_k-$measurable $(k = 0, \cdots, T)$, and likewise for β_k, the amount held in the cash account during the k^{th} period.
The theoretical value of the portfolio at time k is its value right after applying the strategy and is given by

$$V_k = \delta_k S_k + \beta_k, \ k = 1, \cdots, T.$$

We admit only strategies such that each V_k is square-integrable and which replicate the contingent claim H, *i.e.* we require $V_T = H$, which for instance is the case upon choosing $\delta_T = \delta_T^H$ and $\beta_T = \beta_T^H$.
Denote by ΔC_k the incremental cost of applying strategy Φ at time t_k, $k > 0$. In the presence of liquidity costs on the stock, ΔC_k is given by

$$\Delta C_k(\Phi) \quad = \quad \mathcal{L}\left((\delta_k - \delta_{k-1}), S_k, t_k\right) + (\beta_k - \beta_{k-1}) \ \forall k \in \{k = 1, \cdots, T\}$$

where the function $\mathcal{L}$ measures the costs of adjusting the stock part, thereby accounting for the liquidity effect, namely:

- If $(\delta_k - \delta_{k-1}) > 0$, meaning that the strategy requires to buy stock, it might not necessarily be possible at the theoretical price S_k but rather, at a higher price, so that the bigger the quantity to acquire, the greater the marginal costs.

- If on the contrary $(\delta_k - \delta_{k-1}) < 0$, meaning that the strategy now requires to sell, again it might not necessarily be possible at the theoretical price S_k but rather, at a lower price, so that, once again, the bigger the quantity to sell, the greater the marginal costs.

1.2 Liquidity costs

As a consequence of the finite liquidity observed on real markets and described above, it is legitimate to assume that $\mathcal{L} : (\mathbb{R}, \mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}$ is a strictly increasing, convex function of its first argument satisfying $\mathcal{L}(0, ., .) = 0$. Let us make the further assumption that it is differentiable with respect to its first argument, with $\frac{\partial \mathcal{L}}{\partial x}(0, S, .) = S$. At this stage, it is noteworthy to point out that the bid/ask spread is not taken into account, nor is the potential market impact of a transaction. This last assumption amounts to assuming that the period of trading is much greater than the relaxation time of the market impact function.
If there exists an adapted function g, *i.e.* $g = g(x, t, \omega)$ with $\omega \in \mathcal{F}_k$, such that the liquidity costs can be written as $\mathcal{L}((\delta_k - \delta_{k-1}), S_k, t_k) = (\delta_k - \delta_{k-1}) g((\delta_k - \delta_{k-1}), t_k)$, then g is called the supply curve. We refer to [5] for more details on the self-financing approach in the case of a supply curve. Here, a more general assumption is made, namely, that there exists an increasing density function $l : (\mathbb{R}, \mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}$, $l \in \mathcal{C}^1$ representing the cost of buying a marginal amount of stock. That is to say, $\mathcal{L}$ has the following form:

$$\mathcal{L}\left(u, S_k, t_k\right) = \int_0^u l(x, S_k, t_k) dx. \tag{1.1}$$

In particular, $l(0,S_k,t_k)$ is equal to S_k in the absence of bid/ask spread. Assumption (1.1) corresponds to a smoothing of the realistic orderbook profile giving the quantity available at a given price.
In order to make the extension to continuous-time more tractable whilst not narrowing the scope of the paper, we shall further assume that the marginal costs can be written as a stationary function times the theoretical spot price S, *i.e.* $l(x,S,t) = l(x)S_t$.

1.3 Local risk

The local risk is naturally defined, see e.g. [1], as the conditional expectation given information up to time k of the functional associated to the risk function f, of the costs incurred at time $k+1$. This reads

$$\Delta R_k(\Phi) = \mathbb{E}\left(f(\Delta C_{k+1})|\mathcal{F}_k\right)$$

or, with obvious notation,

$$\Delta R_k(\Phi) = \mathbb{E}_k\left(f(\Delta C_{k+1})\right).$$

Note that, contrarily to the case of infinite liquidity, these assumptions on the liquidity costs together with the convexity of the risk function f do not ensure that $(x,y) \mapsto f(\mathcal{L}(x)S+y)$ is a convex function. This lack of convexity will obviously make uniqueness results difficult to obtain.

2 Optimal and pseudo-optimal strategies

The defintion of an optimal strategy is now addressed. Classically, as in [9][1], such a strategy sequentially minimizes the incremental risk process backward in time, and indeed solves the following problem

Problem (*) Given a contingent claim H, find an admissible strategy Φ^* such that

$$\forall k \in (0,\cdots,T-1),\ \forall \Phi \text{ admissible with } \delta_{k+1} = \delta^*_{k+1} \text{ and } \beta_{k+1} = \beta^*_{k+1},\ \Delta R_k(\Phi) \geq \Delta R_k(\Phi^*)$$

Standard regularity and convexity conditions on f, $\mathcal{L}$[1], as well as the assumptions on S_k and β_k, provide the existence of an optimal strategy solution to the first-order optimality equations

$$\left\{\begin{array}{ccc} \mathbb{E}_k\left(f'\left(\Delta C_{k+1}(\Phi^*)\right)\right) & = & 0 \\ \mathbb{E}_k\left(f'\left(\Delta C_{k+1}(\Phi^*)\right)\mathcal{L}'((\delta_{k+1}-\delta_k),S_{k+1},t_{k+1})\right) & = & 0 \end{array}\right.$$

or equivalently

$$\left\{\begin{array}{ccc} \mathbb{E}_k\left(f'\left(\Delta C_{k+1}(\Phi^*)\right)\right) & = & 0 \\ \mathbb{E}_k\left(f'\left(\Delta C_{k+1}(\Phi^*)\right)l\left(\delta_{k+1}-\delta_k\right)S_{k+1}\right) & = & 0 \end{array}\right. \tag{2.1}$$

where $\mathcal{L}'$ stands for the partial derivatives of $\mathcal{L}$ with respect to its first argument.
We now prove the existence of a locally risk-minimizing strategy.

Theorem 2.1 *Problem (*) has a at least one solution Φ^* whose components δ^* and β^* solve the set of equations ((2.1)).*

Proof Let $h(x,y,\omega) \equiv E_k\left(f(\mathcal{L}((U-x),S,t_{k+1}) + (V-y))\right)(\omega)$ with U, V and $S \in \mathcal{L}^2(P)$. We first observe that, thanks to the structure hypotheses on liquidity costs, for a fixed ω, h is a continuous and differentiable function of (x,y) and therefore reaches its minimum at (x^*,y^*) only if (x^*,y^*) is a critical point of h, *i.e.* $\nabla h(x^*,y^*) = 0$. Secondly, there holds that $\lim_{||(x,y)||\to\infty} h(x,y,\omega) = +\infty\ P-a.e.$, so that h has a global

[1] We refer the reader to [1] for the detailed statement of these conditions.

minimum $P-$almost surely. There remains to prove that (x^*, y^*) is $\mathcal{F}_k-$measurable: let $D_n = \{j2^{-n} | j \in \mathbb{Z}\}$ be the set of dyadic rational of order n and define

$$(x_n(\omega), y_n(\omega)) = \inf\{(x,y) \in D_n \times D_n,\ h(x,y,\omega) \leq h(x',y',\omega) \forall (x',y') \in D_n \times D_n\}.$$

Since $\omega \mapsto h(x,y,\omega)$ is $\mathcal{F}_k-$measurable, (x_n, y_n) is also $\mathcal{F}_k-$measurable. As (x_n, y_n) is bounded in n $P-$a.e. and h is continuous in (x,y), $(\tilde{x}, \tilde{y}) = \liminf_{n\to\infty}(x_n, y_n)$ is a $\mathcal{F}_k-$measurable minimizer of h, satisfying $\nabla h(\tilde{x}, \tilde{y}) = 0$. This ends the proof of Theorem 2.1

The set of equations (2.1) can be given a natural interpretation after the introduction of the two processes $C_k^f = \sum_{i=1}^k f'(\Delta C_i)$ and $S_k^S = \sum_{i=1}^k \left(l\left(\Delta\delta_i\right) S_i - l(0) S_{i-1}\right) = \sum_{i=1}^k \left(l\left(\Delta\delta_i\right) S_i - S_{i-1}\right)$ with initial conditions $C_0^f = 0$ and $S_0^S = S_0$: (2.1 simply states that C_k^f is a martingale strongly orthogonal to the martingale part of $\left(S_k^S\right)_k$. The first process will be referred to as the $f-$costs process as in [1], while the new process S^S will be referred to as the supply price process. Following [?], this property of C_k^f will be termed "pseudo-optimality". Let us also mention that, in the case of "infinite" liquidity $l(.) = 1$, the supply price process is just the stock price S, and one recovers the results of [1].

3 Continuous time setting

Let now $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right-continuity and completeness. $T \in \mathbb{R}^{*+}$ denotes a fixed and finite time horizon. Furthermore, we assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$. The risky asset $S = (S_t)_{0\leq t\leq T}$ is supposed to be a strictly positive semimartingale

$$S = S_0 + M + A$$

such that $M = (M_t)_{0\leq t\leq T}$ is a square-integrable martingale with $M_0 = 0$, and $A = (A_t)_{0\leq t\leq T}$ is a continuous and adapted process of finite variation $|A|$ with $A_0 = 0$. Throughout this article, we shall use a right-continuous version of S.

The aim of this section is to define and characterize the f-cost process and the supply process, so as to exend the notions of pseudo-optimality to the continuous-time case. In order to do so, we need to introduce some definitions that will extend the rather intuitive notions of the discrete-time setting to the more intricate continuous-time models.

3.1 Trading strategies and local risk

A general trading strategy Φ is then a pair of càdlàg and adapted processes $\delta = (\delta_t)_{0\leq t\leq T}$, $\beta = (\beta_t)_{0\leq t\leq T}$ while a contingent claim is described by a random variable $H \in L^2(P)$, with $H = \delta^H S_T + \beta^H$, δ^H and β^H being $\mathcal{F}_T-$measurable random variables.

In order to define the processes which are the basic ingredients of pseudo-optimality in continuous time, we need to restrict the set of trading strategies to those we call H-admissible according to the

Definition 3.1.1 *A trading strategy will be called H-admissible if it meets the following requirements*

$$\begin{cases} \delta_T = \delta^H \ P - a.s. \\ \beta_T = \beta^H \ P - a.s. \\ \delta \text{ has finite and integrable quadratic variation} \\ \beta \text{ has finite and integrable quadratic variation} \\ \delta \text{ and } \beta \text{ have finite and integrable quadratic covariation.} \end{cases}$$

We now recap the definitions (see [1]) required to extend the concept of local risk-minimization to the continuous time framework.

Small perturbations

Definition 3.1.2 *A small perturbation is a bounded admissible*[2] *strategy* $\phi = (\beta, \delta)$ *such that* $\beta_T = 0$ *and* $\delta_T = 0$.

Local risk along a partition

Given an $H-$admissible strategy Φ, a partition τ of $[0,T]$, where $\tau = \{0 = t_0, t_1, \cdots, t_k = T\}$ and a small perturbation ϕ, one defines the process

$$r_f^{\tau}[\Phi, \phi](t, \omega) = \sum_{t_i, t_{i+1} \in \tau} \frac{\Delta R_{t_i}(\Phi + \phi|_{[t_i, t_{i+1}(})(\omega) - \Delta R_{t_i}(\Phi)(\omega)}{t_{i+1} - t_i} 1_{[t_i, t_{i+1}(}(t)$$

with $\Delta R_{t_i}(\Phi) = \mathbb{E}\left(f(\Delta C_{t_{i+1}})|\mathcal{F}_{t_i}\right)$ and (as in the previous sections) $\Delta C_{t_{i+1}}$ is the incremental cost associated to a strategy.
The concept of local risk minimization can now be specified:

Definition 3.1.3 *An $H-$admissible strategy Φ is called locally risk-minimizing for the contingent claim H if for every small perturbation ϕ and every increasing sequence of partitions $(\tau_n)_{n \in \mathbb{N}}$ tending to the identity, there holds*

$$\liminf_{n \to \infty} r^{\tau_n}[\Phi, \phi] \geq 0 \ \mathcal{P} - a.e.$$

3.2 The $f-$costs process

Given a general trading strategy Φ, its $f-$cost process $C_t(\Phi)$ [3] is defined as the limit, whenever it exists, of

$$\sum_{k=1}^{l_n} f'(\mathcal{L}(\delta^{\tau_k^n} - \delta^{\tau_{k-1}^n}, S^{\tau_k^n}) + \beta^{\tau_k^n} - \beta^{\tau_{k-1}^n}),$$

where convergence takes place in ucp topology, for any sequences $\mathcal{P}_n$ of Riemann partitions of $[0,T]$ of length l_n (X^T stands for the process stopped at T).

Below is a result showing that the $f-$cost process of an $H-$admissible strategy is well defined.

Proposition 3.1 *The $f-$cost process of an $H-$admissible strategy Φ is well-defined and is given by the formula*

$$\begin{aligned} &C_t(\Phi) = \\ &f''(0)\left(V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s + \tfrac{1}{2} l'(0) \int_{0+}^t S_{s-} d[\delta, \delta]_s^c\right) \\ &+ \tfrac{f^{(3)}(0)}{2}\left([\beta, \beta]_t^c + 2\int_{0+}^t S_{s-} d[\beta, \delta]_s^c + \int_{0+}^t S_{s-}^2 d[\delta, \delta]_t^c\right) \\ &+ \textstyle\sum_{0 < s \leq t} f'(\Delta\beta_s + \mathcal{L}(\Delta\delta_s, S_s)) - f''(0)(\Delta\beta_s + \Delta\delta_s S_s) \end{aligned} \tag{3.1}$$

with notation $[X, Y]^c$ standing for the continuous part of the (càdlàg) quadratic covariation process.

Proof The proof relies on exactly the same ingredients as in Theorem 2 of [1], where the case without transaction costs is thoroughly studied along the lines of the proof of Itō formula's proof for general semimartingales in e.g. [16]. The only (minor) difference lies in the use of Taylor's theorem, which we apply to $f'(\mathcal{L}(x)S + y)$ rather than $f'(S)$.

[2] Admissible means that it satisfies the same regularity requirements as an $H-$admissible strategy without the equality constraints on the terminal conditions.

[3] The superscript f is dropped as the cost function is fixed once and for all and no confusion can occur.

Corollary 3.2 *The $f-$cost process of an $H-$admissible strategy Φ can also be expressed in terms of the portfolio value V*

$$\begin{gathered}
C_t(\Phi) = \\
f''(0)\left(V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s\right) \\
+f''(0)l'(0)\left(\tfrac{1}{2}\int_{0+}^t S_{s-} d[\delta,\delta]_s^c\right) \\
+\tfrac{f^{(3)}(0)}{2}\left([V,V]_t^c - 2\int_{0+}^t \delta_{s-} d[V,S]_s^c + \int_{0+}^t \delta_{s-}^2 d[S,S]_t^c\right) \\
+\textstyle\sum_{0<s\le t} f'(\Delta V_s - \delta_{s-}\Delta S_s + \mathcal{L}(\Delta\delta_s, S_s) - \Delta\delta_s S_s) \\
-\textstyle\sum_{0<s\le t} f''(0)(\Delta V_s - \delta_{s-}\Delta S_s).
\end{gathered} \tag{3.2}$$

Proof A straightforward application of the properties of quadratic variations, when viewing β as a function of V in formula (3.1).

Compared to the case of infinite liquidity, the additional term in the expression of $C_t(\Phi)$ $f''(0)l'(0)\left(\frac{1}{2}\int_{0+}^t S_{s-} d[\delta,\delta]_s^c\right)$, and it is non-decreasing given the convexity of both f and $\mathcal{L}$.

3.3 The supply price process

Exactly as in the previous paragraph, for an $H-$admissible trading strategy Φ, one can define the supply price process $S_t^S(\Phi)$ as the limit in ucp topology, whenever it exists, of

$$\sum_{k=1}^{l_n}\left(l(\delta^{\tau_k^n} - \delta^{\tau_{k-1}^n})S^{\tau_k^n} - S^{\tau_{k-1}^n}\right)$$

for any sequences $\mathcal{P}_n$ of Riemann partitions of $[0,T]$ of length l_n. Given an $H-$admissible strategy Φ, the existence of the associated supply price is ensured by the following result.

Proposition 3.3 *The supply price process S^S of an $H-$admissible strategy Φ is well-defined and given by the formula*

$$\begin{aligned}
S_t^S(\Phi) = S_t + l'(0)\left(\delta_t S_t - \delta_0 S_0 - \int_{0+}^t \delta_{s-} dS_s\right) + \tfrac{1}{2}l''(0)\int_{0+}^t S_{s-} d[\delta,\delta]_s^c \\
+\textstyle\sum_{0<s\le t}\left(l((\Delta\delta_s) - 1)S_s - l'(0)\Delta\delta_s S_s\right).
\end{aligned} \tag{3.3}$$

Proof The proof follows the same lines as that of Proposition 3.1 and is omitted.

4 Application to stochastic volatility models

Of great interest is the particularization of the general concepts previously defined to some specific asset dynamics. In this section, the case of stochastic volatility is considered. In order to derive an explicit formula for the $f-$cost and supply price processes, and completely characterize pseudo-optimal strategies, we then let (S,σ) be a solution of the following set of SDE's

$$dS_t = \mu_t dt + \sigma_t dW_t^1 \tag{4.1}$$

$$d\sigma_t = \gamma_t dt + \Sigma_t dW_t^2 \tag{4.2}$$

where (W^1, W^2) is a two-dimensional Wiener process under $\mathcal{P}$ with correlation ρ, *i.e.* $d < W^1, W^2 >_t = \rho dt$. Under appropriate conditions hold for the functions μ_t, γ_t and Σ_t, see e.g. [**?**], (4.1, 4.2) admits a unique strong continuous solution with $S_t > 0$ and $\sigma_t > 0$. We will from now on assume that such conditions hold true and restrict our study to a Markovian framework, thereby looking for the optimal strategy Φ as a smooth function of the state variables

$$\begin{aligned}
\delta_t &= \delta(t, S_t, \sigma_t) \\
V_t &= V(t, S_t, \sigma_t).
\end{aligned}$$

4.1 PDE formulation

In order to derive a set of PDE's satisfied by pseudo-optimal strategies, one first has to rewrite the cost process as a function of the diffusion parameters and the strategy. A straightforward calculation using (3.2) yields

$$\begin{aligned} C_t(\Phi) = & \int_0^t \Big[f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu_u + \frac{\partial V}{\partial \sigma}\gamma_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_u\mu_u \right) \\ + & \qquad f''(0)l'(0)\tfrac{S_u}{2}\left(\left(\tfrac{\partial \delta}{\partial S}\right)^2 \sigma_u^2 + \left(\tfrac{\partial \delta}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\tfrac{\partial \delta}{\partial S}\tfrac{\partial \delta}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\ + & \qquad \tfrac{f^{(3)}(0)}{2}\left(\left(\tfrac{\partial V}{\partial S}\right)^2 \sigma_u^2 + \left(\tfrac{\partial V}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\tfrac{\partial V}{\partial S}\tfrac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\ - & \qquad f^{(3)}(0)\delta_u\left(\tfrac{\partial V}{\partial S}\sigma_u^2 + \tfrac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u\right) + \tfrac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 \Big] du \\ + & \qquad \int_0^t f''(0)\left(\frac{\partial V}{\partial S} - \delta_u\right)\sigma_u dW_u^1 + \int_0^t f''(0)\frac{\partial V}{\partial \sigma}\Sigma_u dW_u^2. \end{aligned}$$

Likewise, using 3.3), there holds for the supply price process

$$\begin{aligned} S_t^S(\Phi) = & \quad S_t + l'(0)\left(\delta_t S_t - \delta_0 S_0 - \int_0^t \delta_u\mu_u du - \int_0^t \delta_u\sigma_u dW_u^1\right) \\ + & \quad \tfrac{1}{2}l''(0)\int_0^t \left(\left(\frac{\partial \delta}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial \delta}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial \delta}{\partial S}\frac{\partial \delta}{\partial \sigma}\rho\sigma_u\Sigma_u \right) du. \end{aligned}$$

Now, applying to the strategy Φ the first pseudo-optimality criterion, *i.e.* that C must be a martingale under the measure P, we find a first fully non-linear PDE satisfied by the strategy (V, δ)

$$\begin{aligned} f''(0) & \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu_u + \frac{\partial V}{\partial \sigma}\gamma_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_u\mu_u \right) \\ & + f''(0)l'(0)\frac{S_u}{2}\left(\left(\frac{\partial \delta}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial \delta}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial \delta}{\partial S}\frac{\partial \delta}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\ & + \frac{f^{(3)}(0)}{2}\left(\left(\frac{\partial V}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial V}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\ & - f^{(3)}(0)\delta_u\left(\frac{\partial V}{\partial S}\sigma_u^2 + \frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u\right) + \frac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 \quad = 0. \end{aligned}$$

with terminal condition corresponding $V_T = \delta^H S_T + \beta^H$.
In order to apply the second pseudo-optimality criterion, *i.e.* that the martingale C must be orthogonal to the martingale part of the supply price process S^S, we first identify the martingale part of the latter

$$S_t^S(\Phi) - \mathbb{E}\left(S_t^S(\Phi)\right) = \quad \int_0^t \left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma_u dW_u^1 + \int_0^t l'(0)S\frac{\partial \delta}{\partial \sigma}\Sigma_u dW_u^2,$$

so that the second PDE satisfied by (V, δ) is

$$\begin{aligned} \left(\frac{\partial V}{\partial S} - \delta\right)\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma^2 + \frac{\partial V}{\partial \sigma}\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\rho\sigma\Sigma \quad + \\ \left(\frac{\partial V}{\partial S} - \delta\right)\frac{\partial \delta}{\partial \sigma}l'(0)S\rho\sigma\Sigma + \frac{\partial V}{\partial \sigma}\frac{\partial \delta}{\partial \sigma}l'(0)S\Sigma^2 \quad = 0. \end{aligned}$$

With some rearrangements, the pseudo-optimal strategy Φ is shown to solve the following coupled system of nonlinear PDEs

$$\begin{cases} \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu + \frac{\partial V}{\partial \sigma}\gamma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma\Sigma = \\ \delta\mu + \alpha\left(\left(\frac{\partial V}{\partial S}\sigma + \frac{\partial V}{\partial \sigma}\rho\Sigma - \delta\sigma\right)^2 + (1-\rho^2)\left(\frac{\partial V}{\partial \sigma}\right)^2\Sigma^2\right) \\ \quad + l'(0)\frac{S}{2}\left(\left(\frac{\partial \delta}{\partial S}\sigma + \frac{\partial \delta}{\partial \sigma}\rho\Sigma\right)^2 + (1-\rho^2)\left(\frac{\partial \delta}{\partial \sigma}\right)^2\Sigma^2\right) \\ \left(\frac{\partial V}{\partial S} - \delta\right)\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma^2 + \frac{\partial V}{\partial \sigma}\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\rho\sigma\Sigma \\ \quad + \left(\frac{\partial V}{\partial S} - \delta\right)\frac{\partial \delta}{\partial \sigma}l'(0)S\rho\sigma\Sigma + \frac{\partial V}{\partial \sigma}\frac{\partial \delta}{\partial \sigma}l'(0)S\Sigma^2 = 0 \end{cases} \tag{4.3}$$

with terminal condition $V_T = \delta^H S_T + \beta^H$.
A system such as (4.3) is quite challenging: one can see it as a parabolic equation coupled with a nonlinear, stationary hyperbolic equation which can be viewed as a constraint. The study of (4.3) will be the subject of another work.

The case of a complete market

The case of a complete market corresponds to Σ, the volatility of volatility, equal to zero. The equation for the hedge ratio δ then reduces to

$$\left(\frac{\partial V}{\partial S} - \delta\right)\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right) = 0.$$

so that a sufficient condition is that V, δ is a solution to

$$\delta = \frac{\partial V}{\partial S} \tag{4.4}$$

$$\frac{\partial V}{\partial u} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\left(1 + l'(0)S\frac{\partial^2 V}{\partial S^2}\right) = 0. \tag{4.5}$$

Upon the generalized Black and Scholes PDE (4.5) having a solution, a property easily shown to hold when the contingent claim has a convex payoff, equation (4.4) gives a perfect hedge. As already holds in the infinite liquidity case, the solution does not depend on the risk function f. In fact, the $f-$cost process is identically zero, which amounts to having a self-financing strategy incorporating liquidity costs that perfectly replicates the contingent claim H.

4.2 The minimization problem

Despite the fact that, in discrete time, a pseudo-optimal strategy satisfying ((2.1)) might not be optimal, in continuous time, and when working with continuous path processes, there exists a correspondence between the two concepts [4]. As a matter of fact, we now prove that a strategy solving system of equations (4.3) is locally risk-minimizing for the function f.
Given the smoothness of the risk function f and the liquidity costs function $\mathcal{L}$, one can write a Taylor expansion around the unperturbed strategy Φ. Given a partition τ and $t \in [0, T]$, and assuming without loss of generality that t is one of the t^i, there holds that

$$\begin{aligned} r_f^\tau[\Phi, \phi](t, \omega) &= \frac{\Delta R_{t_i}(\Phi + \phi|_{[t_i, t_{i+1}(})(\omega) - \Delta R_{t_i}(\Phi)(\omega)}{t_{i+1} - t_i} \\ &= \frac{\mathbb{E}_{t_i}\left(f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i, t_{i+1}(}))\right)(\omega) - \mathbb{E}_{t_i}\left(f(\Delta C_{t_{i+1}}(\Phi))\right)(\omega)}{t_{i+1} - t_i} \end{aligned}$$

[4]We refer the reader to paragraph 3.2 of [1] for the precise definition of local risk-minimization in continuous time.

Taylor's formula applied to $g : (x,y) \mapsto f(\mathcal{L}(x)+y)$ then yields for some $\tilde{\phi} \in []$

$$\begin{aligned} &f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i,t_{i+1}()})) = \\ f(\Delta C_{t_{i+1}}(\Phi)) - \beta_{t_i} f'(\Delta C_{t_{i+1}}(\Phi)) - \delta_{t_i}\mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi)) + \frac{1}{2}\delta_{t_i}^2 h(\tilde{\phi}) \\ + \frac{1}{2}(\beta_{t_i} + \delta_{t_i}\mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi}) \end{aligned}$$

where $g(\tilde{\phi}) = f''(\Delta C_{t_{i+1}}(\tilde{\phi}))$ and $h(\tilde{\phi}) = \mathcal{L}''(\tilde{\phi}) f'(\Delta C_{t_{i+1}}(\tilde{\phi}))$ with $\tilde{\phi} = (\tilde{\beta}, \tilde{\delta})$ such that $|\tilde{\beta}| \leq \beta$ and $|\tilde{\delta}| \leq \delta$. Using the standing assumptions on f, namely $f'(0) = 0$, $f''(0) > 0$, the remainder term $\delta_{t_i}^2 h(\tilde{\phi}) + (\beta_{t_i} + \delta_{t_i}\mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi})$ will remain strictly positive in a neighborhood of t_i for δ_{t_i} and β_{t_i} small enough. Rearranging and simplifying yields

$$\begin{aligned} r_f^\tau[\Phi,\phi](t,\omega) &= \beta_{t_i}\frac{\mathbb{E}_{t_i}\left(f'(\Delta C_{t_{i+1}}(\Phi))\right)(\omega)}{t_{i+1}-t_i} + \delta_{t_i}\frac{\mathbb{E}_{t_i}\left(\mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi))\right)(\omega)}{t_{i+1}-t_i} && (4.6) \\ &+ \frac{1}{2}\frac{\mathbb{E}_{t_i}\left(\delta_{t_i}^2 h(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i} && (4.7) \\ &+ \frac{1}{2}\frac{\mathbb{E}_{t_i}\left((\beta_{t_i} + \delta_{t_i}\mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i}. && (4.8) \end{aligned}$$

Thanks to the pathwise conitnuity of Itō processes, there holds

$$\begin{aligned} \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(f'(\Delta C_{t_{i+1}}(\Phi))\right)(\omega)}{t_{i+1}-t_i} &= \Lambda\left(f' \circ \Delta C\right)_{t_i} \\ \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(\mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi))\right)(\omega)}{t_{i+1}-t_i} &= \Lambda\left(\mathcal{L}' \cdot f' \circ \Delta C\right)_{t_i} \end{aligned}$$

and

$$\begin{aligned} \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(h(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i} &= \Lambda h_{t_i} \\ \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(g(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i} &= \Lambda g_{t_i} \\ \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(\mathcal{L}' g(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i} &= \Lambda\left(\mathcal{L}' \cdot g\right)_{t_i} \\ \lim_{t_{i+1}\to t_i} \frac{\mathbb{E}_{t_i}\left(\mathcal{L}'^2 g(\tilde{\phi})\right)(\omega)}{t_{i+1}-t_i} &= \Lambda\left(\mathcal{L}'^2 \cdot g\right)_{t_i} \end{aligned}$$

where Λ is the infinitesimal generator associated with the diffusion:

$$\Lambda h = \frac{\partial h}{\partial S}\mu + \frac{\partial h}{\partial \sigma}\gamma + \frac{1}{2}\frac{\partial^2 h}{\partial S^2}\sigma^2 + \frac{1}{2}\frac{\partial^2 h}{\partial \sigma^2}\Sigma^2 + \frac{\partial^2 h}{\partial S \partial \sigma}\rho\sigma\Sigma.$$

Finally, one obtains that the process r_f^τ in (4.6) is given by

$$\begin{aligned} r_f^\tau[\phi,\Delta](t,\omega) = &\quad \beta_t \Lambda\left(f' \circ \Delta C\right)_t + \delta_t \Lambda\left(\mathcal{L}' \cdot f' \circ \Delta C\right)_t && (4.9) \\ + &\quad \tfrac{1}{2}\left(\beta_t^2 \Lambda g_t + 2\beta_t\delta_t\Lambda\left(\mathcal{L}' \cdot g\right)_t + \delta_t^2 \Lambda\left(\mathcal{L}'^2 \cdot g + h\right)_t\right). && (4.10) \end{aligned}$$

Upon setting the first component δ of the perturbation equal to zero (that is, we perturb only β), a first condition for a strategy ϕ to be locally risk-minimizing is derived:

$$\beta_t \Lambda \left(f' \circ \Delta C\right)_t + \frac{1}{2}\beta_t^2 \Lambda g_t \geq 0 \; P-a.e. \; \forall \beta_t.$$

Hence, there holds $\Lambda \left(f' \circ \Delta C\right)_t = 0$. Similarly, upon setting now $\beta = 0$, the following second condition for the strategy ϕ follows:

$$\delta_t \Lambda \left(\mathcal{L}' \cdot f' \circ \Delta C\right)_t + \frac{1}{2}\delta_t^2 \Lambda \left(\mathcal{L}'^2 \cdot g + h\right)_t \geq 0 \; P-a.e. \; \forall \delta_t.$$

As a consequence, $\Lambda \left(\mathcal{L}' \cdot f' \circ \Delta C\right)_t = 0$, and one can easily workout the equivalence below

$$\left\{ \begin{array}{rcl} \Lambda \left(f' \circ \Delta C\right)_t & = & 0, \\ \Lambda \left(\mathcal{L}' \cdot f' \circ \Delta C\right)_t & = & 0 \end{array} \right.$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{rcl} f''(0)\left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu_u + \frac{\partial V}{\partial \sigma}\gamma_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_u\mu_u\right) & & \\ +f''(0)l'(0)\frac{S_u}{2}\left(\left(\frac{\partial \delta}{\partial S}\right)^2\sigma_u^2 + \left(\frac{\partial \delta}{\partial \sigma}\right)^2\Sigma_u^2 + 2\frac{\partial \delta}{\partial S}\frac{\partial \delta}{\partial \sigma}\rho\sigma_u\Sigma_u\right) & & \\ +\frac{f^{(3)}(0)}{2}\left(\left(\frac{\partial V}{\partial S}\right)^2\sigma_u^2 + \left(\frac{\partial V}{\partial \sigma}\right)^2\Sigma_u^2 + 2\frac{\partial V}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u\right) & & \\ -f^{(3)}(0)\delta_u\left(\frac{\partial V}{\partial S}\sigma_u^2 + \frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u\right) + \frac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 & = & 0, \\ \left(\frac{\partial V}{\partial S} - \delta\right)\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma^2 + \frac{\partial V}{\partial \sigma}\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\rho\sigma\Sigma & & \\ +\left(\frac{\partial V}{\partial S} - \delta\right)\frac{\partial \delta}{\partial \sigma}l'(0)S\rho\sigma\Sigma + \frac{\partial V}{\partial \sigma}\frac{\partial \delta}{\partial \sigma}l'(0)S\Sigma^2 & = & 0. \end{array} \right.$$

As claimed in the beginning of this section, one can see that the optimal strategies with respect to local risk-minimization are the same as the pseudo-optimal strategies. This result is similar to that obtained in [1] in the case of "infinite" liquidity, the only requirement being that the infinitesimal generator is a local operator.

5 Application to stochastic volatility/jump diffusion models

This section is devoted to a situation where non-quadratic risk definitely implies a different hedging strategy. The evolution of S is modelled by an SDE with stochastic volatility and Poisson jumps in the vein of the Bates model [**?**]

$$\begin{array}{rcl} dS_t & = & \mu_t dt + \sigma_t dW_t^1 + k dN_t \\ d\sigma_t & = & \gamma_t dt + \Sigma_t dW_t^2, \end{array}$$

where as before W^1 and W^2 are Wiener processes under $\mathcal{P}$ and $d < W^1, W^2 >_t = \rho dt$, N_t is a Poisson process with intensity λ, and the amplitude of the jumps k has probability distribution K. We also assume that W_t, N_t and k are independent. Similarly to the case of stochastic volatility, standard assumptions are made to ensure that the set of SDE has a unique strong solution.
With these assumptions, one can look for the optimal strategy Φ, in a Markovian framework, as a function of the state variables

$$\begin{array}{rcl} \delta_t & = & \delta(t, S_t, \sigma_t) \\ V_t & = & V(t, S_t, \sigma_t) \end{array}$$

and derive an equation for these quantities.

5.1 PIDE formulation

The PIDE's corresponding to respectively the portfolio value and optimal strategy follow as in Section **??** fro a rewriteing of the cost process

$$\begin{aligned}
C_t(\Phi) = \quad & \int_0^t \Big(f''(0) \left(\tfrac{\partial V}{\partial u} + \tfrac{\partial V}{\partial S}\mu_u + \tfrac{\partial V}{\partial \sigma}\gamma_u + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma_u^2 + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \tfrac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_u\mu_u \right) \\
+ \quad & \tfrac{f^{(3)}(0)}{2} \left(\left(\tfrac{\partial V}{\partial S}\right)^2 \sigma_u^2 + \left(\tfrac{\partial V}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\tfrac{\partial V}{\partial S}\tfrac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\
+ \quad & \tfrac{l'(0)S_u}{2} \left(\left(\tfrac{\partial \delta}{\partial S}\right)^2 \sigma_u^2 + \left(\tfrac{\partial \delta}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\tfrac{\partial \delta}{\partial S}\tfrac{\partial \delta}{\partial \sigma}\rho\sigma_u\Sigma_u \right) \\
- \quad & f^{(3)}(0)\delta_{u-} \left(\tfrac{\partial V}{\partial S}\sigma_u^2 + \tfrac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) + \tfrac{f^{(3)}(0)}{2}\delta_u^2\sigma_u^2 \Big) \, du \\
+ \quad & \int_0^t f''(0) \left(\tfrac{\partial V}{\partial S} - \delta_{u-} \right) \sigma_u dW_u^1 \\
+ \quad & \int_0^t f''(0) \tfrac{\partial V}{\partial \sigma} \Sigma_u dW_u^2 \\
+ \quad & \int_0^t \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-}\Delta S_u + \mathcal{L}(\Delta\delta_u, S_u) - \Delta\delta_u S_u) K(k) dk dN_u
\end{aligned}$$

which follows from equation (3.2). Note that ΔV_u, the jump of V when there S has a jump ΔS_u of size k at time u, is equal to $V(u-, S_{u-} + k, \sigma_{u-}) - V(u-, S_{u-}, \sigma_{u-})$, and similarly for $\Delta\delta_u$.
Applying the first pseudo-optimality criterion to the strategy Φ, *i.e.* that C is a martingale under the measure P, yields the PIDE satisfied by the portfolio value V

$$\begin{aligned}
f''(0) \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial S}\mu_u + \frac{\partial V}{\partial \sigma}\gamma_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_u^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\rho\sigma_u\Sigma_u - \delta_{u-}\mu_u \right) & \\
+ \frac{f^{(3)}(0)}{2} \left(\left(\frac{\partial V}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial V}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial V}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) & \\
+ \frac{l'(0)S_u}{2} \left(\left(\frac{\partial \delta}{\partial S}\right)^2 \sigma_u^2 + \left(\frac{\partial \delta}{\partial \sigma}\right)^2 \Sigma_u^2 + 2\frac{\partial \delta}{\partial S}\frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) & \\
- f^{(3)}(0)\delta_{u-} \left(\frac{\partial V}{\partial S}\sigma_u^2 + \frac{\partial V}{\partial \sigma}\rho\sigma_u\Sigma_u \right) + \frac{f^{(3)}(0)}{2}\delta_{u-}^2\sigma_u^2 & \\
+ \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-}\Delta S_u) K(k) dk \lambda_u & = 0
\end{aligned}$$

with terminal conditions

$$V_T = \delta^H S_T + \beta^H.$$

In order to apply the second pseudo-optimality criterion, *i.e.* that the martingale C be orthogonal to the martingale part of the supply price process S^S, we first identify the martingale part

$$\begin{aligned}
S_t^S(\Phi) - \mathbb{E}\left(S_t^S(\Phi)\right) &= \int_0^t \left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma_u dW_u^1 + \int_0^t l'(0)S\frac{\partial \delta}{\partial \sigma}\Sigma_u dW_u^2 \\
& \quad + \int_0^t \int_{\mathbb{R}} \left((l(\Delta\delta_u) - 1)S_u + k\right) K(k) dk d\tilde{N}_u
\end{aligned}$$

with $\tilde{N}$, the compensated Poisson process of N. Therefore the second PDE satisfied by the strategy (V, δ) is

$$\begin{aligned}
\left(\frac{\partial V}{\partial S} - \delta\right)\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\sigma^2 + \frac{\partial V}{\partial \sigma}\left(1 + l'(0)S\frac{\partial \delta}{\partial S}\right)\rho\sigma\Sigma & \\
+ \left(\frac{\partial V}{\partial S} - \delta\right)\frac{\partial \delta}{\partial \sigma}l'(0)S\rho\sigma\Sigma + \frac{\partial V}{\partial \sigma}\frac{\partial \delta}{\partial \sigma}l'(0)S\Sigma^2 & \\
+ \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-}\Delta S_u + \mathcal{L}(\Delta\delta_u, S_u) - \Delta\delta_u S_u)\left((l(\Delta\delta_u) - 1)S_u + k\right) K(k) dk \lambda_u & = 0
\end{aligned}$$

again with terminal conditions $V_T = \delta^H S_T + \beta^H$.
Contrarily to the stochastic volatility case, the optimal strategy in the jump-diffusion model requires the knowledge of both functions f and $\mathcal{L}$ on their whole domain of definition. Thas feature was to be expected from the non-local character of the associated infinitesimal generator.

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